A NOTE ON MIXING TRANSFORMATIONS

BY

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ABSTRACT

Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and T a 1-1, onto, measure-preserving transformation. Necessary and sufficient conditions are given for T to be mixing, in terms of union of iterates of sets.

1. Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and τ a 1-1, onto, bimeasurable and measure-preserving transformation. τ is ergodic if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n-1} \mu(\tau^j A \cap B) = \mu(A)\mu(B)$$

for all A and $B \in \mathscr{F}$, and τ is mixing if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n-1} \left| \mu(\tau^j A \cap B - \mu(A)\mu(B) \right| = 0$$

for all such sets. From now on all sets mentioned will be assumed to be in $\mathscr{F}$.

Let k be an infinite sequence of positive integers and for a set A define $A(k) = \bigcup_{k_n \in k} \tau^{k_n} A$. It is well known that τ is ergodic if and only if $\mu[A(k)] = 1$ for all positive A [i.e., $\mu(A) > 0$], and k the sequence of all positive integers. From this it follows at once that τ is completely ergodic, i.e. every power of τ is ergodic, if and only if $\mu[A(k)] = 1$ for every positive A and every arithmetic sequence k. It is therefore of some interest to find out how much this condition must be strengthened to obtain mixing, and we do so in this note.

We shall say k is a (1,2)-sequence if $k_{n+1} - k_n = 1$ or 2, and whenever $k_{n+1} - k_n = 2$ then $k_{n+2} - k_{n+1} = 1$. We shall use the customary definitions of asymptotic (upper, lower) density of a sequence.

THEOREM. *Let τ be completely ergodic. Then τ is mixing if and only if there*

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exists $\varepsilon > 0$ *such that* $\mu[A(k)] = 1$ *for every* $(1,2)$*-sequence of asymptotic lower density* $\geqq 1 - \varepsilon$, *and every positive* A.

The necessity of the condition follows at once since if τ is mixing then $\lim_n \mu(\tau^n A \cap B) = \mu(A)\mu(B)$ for every A and B, except possibly along a sequence k which may depend on A and B, but has asymptotic density zero for every A and B. Thus if k has positive asymptotic lower density and for some positive A we have $\mu[A(k)] < 1$ then we let $B = (A(k))^c$ to obtain a contradiction.

For the sufficiency we shall need to introduce the spectrum of τ. By this we mean the spectrum of the unitary operator U defined on $L_2(\Omega, \mu)$ and given by $Uf(x) = f(\tau x)$ for $f \in L_2$. Now it is well known that τ is completely ergodic if and only if every eigenvalue λ of U, distinct from 1, is not a root of unity; and moreover τ is mixing if and only if U has no eigenvalues distinct from 1. Thus let us suppose to the contrary that U has an eigenvalue λ which is not a root of unity. If f is the associated eigenfunction it is also known that $|f|$ is a constant, provided τ is ergodic. We shall have occasion to use the following lemma which has some independent interest.

LEMMA. *Let* $|f| = 1$, *and let* B *be any Borel subset of the unit circle. Then* $\mu[f^{-1}(B)] = m(B)$, *where* m *is normalized Lebesgue measure on the unit circle.*

PROOF. Assume first that B is a subinterval of the circle and let $A = f^{-1}(B)$. Then $\lim_{n\to\infty} 1/n \sum_{j=1}^{n-1} \chi_A(\tau^j x) = \mu(A)$ on a set of μ measure one, where χ_A is the set characteristic function of A. Let x be in this set and note that $\tau_j x \in A$ if and only if $f(\tau^j x) = \lambda^j f(x) \in B$. Thus $1/n \sum_{j=1}^{n-1} \chi_A(\tau^j x)$ is also the proportion of times that $\lambda^j f(x)$ is in B. But from the Weyl equidistribution theorem it follows that this proportion converges to $m(B)$. Once we have this for single interval, it is also true for all intervals with rational endpoints on the same set of μ measure one, and therefore for all intervals. The rest follows from standard arguments.

To complete the proof of the theorem, we shall assume that $\lambda = e^{i\theta}$ is small in the sense that $0 < \theta \leqq \pi/20$, say. We can do this since if λ is an eigenvalue of U, so is λ^n for all integers n. Suppose that $\varepsilon > 0$ is given and that δ is so chosen that $m\{[1, e^{i\delta}]\} < \varepsilon/2$ and also $0 < \delta < \theta/20$. Let $I = [1, e^{i\delta}]$, and $A = f^{-1}(I)$. Note that $f(\tau^k A) = [\lambda^k, \lambda^k e^{i\delta}]$. Define the sequence k by choosing $m \in k$ providing $f(\tau^m A) \cap I = \varnothing$, or equivalently provided $A \cap \tau^m A = \varnothing$. Then clearly k is a $(1,2)$-sequence and the density condition is easily verified. On the other hand $\mu(A) = m(I) > 0$ and $A(k) \cap A = \varnothing$. Thus $\mu[A(k)] < 1$, and the theorem is proved.

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